Lecture 24 Notes

April 21, 2011

ITT82 Proof of the Axiom of Choice in Refinement version

Assume A type, $B : A \to \text{Type}$, $C : x{:}A \to B(x) \to \text{Prop}$,

$\vdash \forall x{:}A.\ \exists y{:}B(x).\ C(x,y) \supset \exists f{:}(x{:}A \to B(x)).\ \forall x{:}A.\ C(x, f(x))$

by $\lambda(z.\ \_\_\_)$

$z : \forall x{:}A.\ \exists y{:}B(x).\ C(x,y)$

$\vdash \exists f{:}(x{:}A \to B(x)).\ \forall x{:}A.\ C(x,f(x))$

by $\langle$ slot-f, slot-pf $\rangle$

$\vdash x{:}A \to B(x)$ by $\lambda(x.\ spread(z(x); x,y.x))$
call this f { Martin-Löf calls it $\lambda(x.\ p(z(x)))$. }

Note $x{:}A \to B(x)$ is a type based on the assumptions about A, B(x). In CTT this is proved by well-formedness rules interleaved with the proof. In ITT82 this is established before the logical proof.

$\vdash \forall x{:}A.\ C(x, f(x))$ by $\lambda(x.\ \_\_\_)$

$x{:}A \vdash C(x, f(x))$ by $apseq(z; x; v.\ \_\_\_)$

$x{:}A,\ v = z(x),\ v : \exists y{:}B(x).\ C(x,y)$ { This is Martin-Löf's first step, page 173. }

$\vdash C(x, f(x))$ by $spread(z(x); x,y.y)$

Note, $C(x,f(x))$ is a type based on the assumptions about A, B(x).
Also note, $spread(z(x); x,y.x) \in B(x)$. Martin-Löf calls this $p(z(x))$ using his definitions on page 174 and proves $q(z(x)) \in C(x, p(z(x)))$ where $q(z(x)) = spread(z(x); x,y.y)$. His proof builds up from $p(z(x))$ and $q(z(x))$ by defining

$\lambda(x.\ p(z(x)))$, our function f.